Chapter 7 - Vectors and Cartesian Geometry*

Contents

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1 Scalars and Vectors

As you may already know, a vector is in a certain sense a collection of numbers, sometimes arranged in the form of a column or row. This is where it is important to be well aware of the difference between vectors, these collections of numbers organised like matrices, and scalars, the numbers that make up these arrangements.

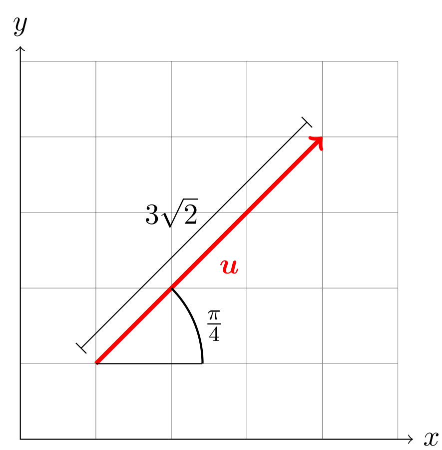
1.1 Definitions

Definition 1.1. A **scalar** is a quantity whose only characteristic is *magnitude* (size). For example, mass, area, speed and distance are scalars.

A **vector** is a quantity whose characteristics are *magnitude* and *direction*. For example, displacement, velocity and acceleration are all vector quantities.

Quite often, vectors arise in geometry, because they can be represented by arrows of a certain length (magnitude) with a certain orientation (direction).

Example 1.2. In the following xy-plane, $\boldsymbol{u}$ is a vector with magnitude $3\sqrt{2}$, and whose direction is at a $\frac{\pi}{4}$ angle with the x-axis. You could think of this vector as representing a force, for example.



In this case, rather than defining $\boldsymbol{u}$ by its direction and magnitude, it may be easier to define it using x and y coordinates. If the xy-plane were a map and you were trying to follow the path determined by the direction and magnitude of $\boldsymbol{u}$, you would take 3 steps in the direction of x and 3 steps in the direction of y - so the coordinates of $\boldsymbol{u}$ are $(3, 3)$.

To see how to express a 2D vector using x and y coordinates, see this interactive graph.

1.2 Vectors in Coordinate Geometry

Rather than saying that a vector has direction and magnitude, we quite often prefer to say that a vector is a sequence of numbers. In 2D, like in Example 1.2, one only needs two coordinates - here, x and

y - to describe a vector. In 3D, an additional coordinate is needed, such as z, to account for the additional dimension. If we were to go into higher dimensions, we'd realise that to describe a vector in n dimensions, we need n coordinates. This brings about a different definition of a vector:

Definition 1.3. If n is a positive integer, an n-dimensional **vector** is an n-tuple (an sequence of n numbers), usually written as follows:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \quad \text{or} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

While these two ways of writing a vector seem similar, they are slightly different - although both the column and row notations exist, writing vectors in columns is generally more accepted and leads to less confusion. However, we will sometimes also use the row version in this document for stylistic reasons.

This tuple of numbers uniquely determines a vector which has its own direction and magnitude - so the two definitions we have seen (Definition 1.1 and Definition 1.3) are two equivalent ways of viewing a vector.

In practice, we will generally stick to vectors in three dimensions at most, and so this definition simplifies down to:

$$\boldsymbol{x} = (x_1, x_2, x_3) \quad \text{or} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{1}$$

Usually, on typeset documents, vectors are referred to using a bold italic letter. In handwriting, to avoid the difficulties of making your handwriting look thicker for certain letters, you can write a vector either by underlining the letter, or by putting an arrow on top of it: so $\boldsymbol{x} = \underline{x} = \vec{x}$ all mean the same thing.

Before we can represent vectors in 3D space, we need to give this space a **frame of reference**. This will define what a distance of 1 corresponds to, and will show what directions x, y and z point in. To construct a frame of reference, we need:

- An **origin** O, a point which we decide is the centre of our space;
- Three **unit vectors**, which are all of magnitude 1 and which point along three perpendicular straight lines, corresponding to the usual x, y and z axes. We call these unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively. Each unit vector needs to be perpendicular to the other two, and the three vectors need to follow something called the *right-hand rule*: this says that if you draw positive x pointing right and positive y pointing up on a piece of paper, then positive z needs to point out of the paper, not into the paper - see Figure 1.1.

The frame of reference we have constructed corresponds to **Cartesian coordinates**. However, there

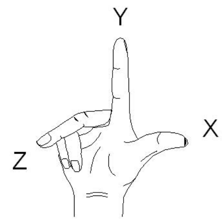


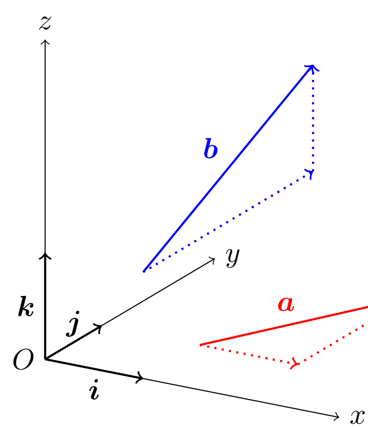
Figure 1.1: Illustration of the Right-Hand Rule

exist other coordinate systems which are also widely used, such as cylindrical polar coordinates and spherical coordinates.

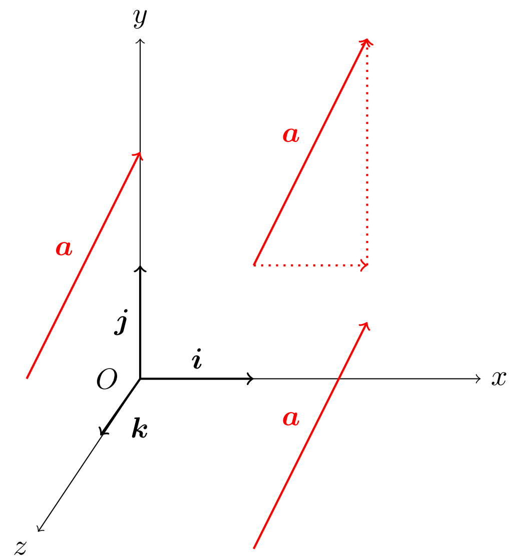
In this coordinate system, our three unit vectors have coordinates:

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

Example 1.4. In the following 3D frame of reference, the vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ have been represented, as well as the vectors $\boldsymbol{a} = (1, 2, 0)$ and $\boldsymbol{b} = (0, 3, 1)$.



One thing that you may have noticed when looking at this example is that vectors have direction and magnitude, but when you represent them in space, they don't have a specific *location*. The vector $\boldsymbol{a}$ in the example goes from the point $(1, 1, 0)$ to the point $(2, 3, 0)$ but we can represent $\boldsymbol{a}$ elsewhere without changing it. In the following image, the three vectors represented are all equal to $\boldsymbol{a} = (1, 2, 0)$, but have different origins; we have also rotated the view, so that we are now looking at the previous diagram from above.



The basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ from our frame of reference can also be used to decompose a vector into parts. Looking back to our vector $\boldsymbol{b} = (0, 3, 1)$ from Example 1.4, we managed to break it down into two different vectors: firstly, a vector going three steps upwards, and then a vector going one step in the direction of z. But one step upwards is the same as $\boldsymbol{j}$, and one step in the direction of z is $\boldsymbol{k}$: so actually, the vector $\boldsymbol{b}$ is just three lots of $\boldsymbol{j}$, plus $\boldsymbol{k}$. We can write this as:

$$\boldsymbol{b} = (0, 3, 1) = 3\boldsymbol{j} + \boldsymbol{k}.$$

This decomposition works for any vector in a 3D frame of reference, as follows:

Theorem 1.5. *Let $O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ define a frame of reference in space, and let $\boldsymbol{x}$ be a vector in this space with coordinates (x_1, x_2, x_3). Then we can write:*

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}.$$

This is called a *linear combination* of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. Multiplying vectors by scalars to get things such as $x\boldsymbol{i}$, like in the theorem, is covered in Subsection 2.1.

One extremely useful property of vector coordinates is that they are **unique**: there is only one way of breaking down $\boldsymbol{x}$ into $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

Theorem 1.6 (Uniqueness of Vector Coordinates)**.** *Given two vectors* $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ *and* $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$*, we have:*

$$\boldsymbol{u} = \boldsymbol{v} \iff \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \iff \begin{cases} u_1 = v_1 \\ u_2 = v_2 \\ u_3 = v_3. \end{cases}$$

In other words, two vectors are equal if and only if their coordinates are equal.

One final thing to note before moving on to the next topic is the existence of a vector called the **zero vector**. This vector is a sort of analogue of the number 0, but for vectors. We usually denote it by a bold zero: $\mathbf{0}$. In 3D, its coordinates are $(0, 0, 0)$. In other dimensions, it's almost the same thing: $(0, 0)$ in 2D, $(0, 0, 0, 0)$ in 4D, and so on. Once again, when writing this down by hand it is common to write $\underline{0}$ or $\vec{0}$.

1.3 Position Vectors

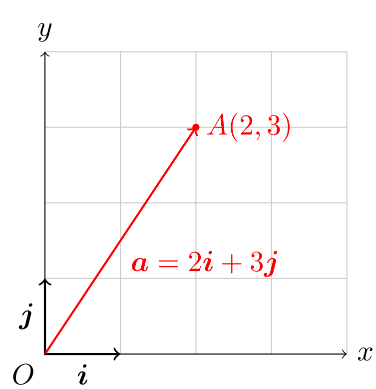
Given a point in space, a vector can be used to describe its position - this vector is called a **position vector**. Let's suppose you have a point A in 3D space. This point A has coordinates (a_1, a_2, a_3) which describe its position - we sometimes write A as $A(a_1, a_2, a_3)$ to indicate these coordinates. If you take the coordinates of A to represent a vector rather than a point, then you obtain the position vector of this point:

Definition 1.7. The **position vector** of a point A with coordinates (a_1, a_2, a_3) is the vector:

$$\boldsymbol{a} = (x, y, z) = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}.$$

If you follow the direction that this vector indicates, starting from the origin $O(0, 0, 0)$, you will end up at the point A.

Example 1.8. We will consider an example in the xy-plane for simplicity. Here we have a point A with coordinates $(2, 3)$.



The vector $\boldsymbol{a}$ could be represented anywhere, but representing it here with the base at the origin O points us to the position of A.

From this, you might be able to guess what the position vector of the origin O is: since the vector leading you from O to O takes you nowhere, the corresponding position vector is the zero vector $\mathbf{0}$.

2 Operations on Vectors

We will suppose in this section that our vectors are in 3D space and therefore have coordinates (x_1, x_2, x_3) where x_1, x_2 and x_3 are real numbers. However, most of the concepts presented here can be generalised to lower and, indeed, higher dimensions. Remember that vectors are not always geometric objects!

2.1 Basic Operations

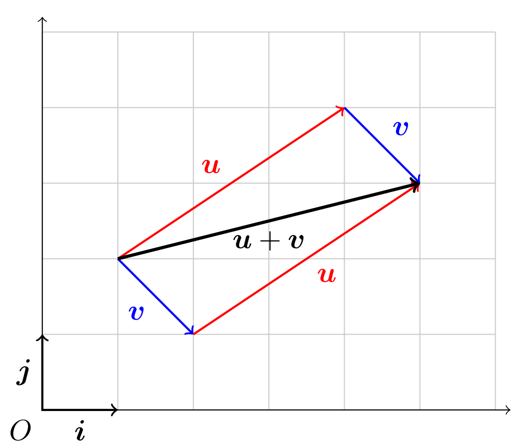
The first operation to consider is addition. In the case of vectors, addition is quite simple and can be done **as long as the vectors have the same number of entries**. Addition is then *componentwise*, which means we add up the coordinates one by one as follows:

Definition 2.1. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Then the sum of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is:

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}.$$

Example 2.2. *In the xy-plane, let $\boldsymbol{u} = (3, 2)$ and $\boldsymbol{v} = (1, -1)$. Find $\boldsymbol{u} + \boldsymbol{v}$ and show geometrically that $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.*

Answer. From the rules of addition outlined above in Definition 2.1, $\boldsymbol{u} + \boldsymbol{v} = (3, 2) + (1, -1) = (3 + 1, 2 - 1) = (4, 1)$. In the plane, this is:



From this picture we can see that whether we follow $\boldsymbol{u}$ and then $\boldsymbol{v}$, or $\boldsymbol{v}$ and then $\boldsymbol{u}$, we end up at the same point: this shows that $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$, which is a property called *commutativity* (you need not remember this). Because the drawing forms a parallelogram, we call this property the "parallelogram law".

You can try out addition with other vectors on this interactive graph.

Another natural operation to consider is multiplication by scalars. Let's suppose you have a real number λ, and a vector $\boldsymbol{u}$. Then the vector $\boldsymbol{u}$ can be "scaled" componentwise, like in the following definition:

Definition 2.3. The product of a vector $\boldsymbol{u} = (u_1, u_2, u_3)$ with a scalar λ is:

$$\lambda \boldsymbol{u} = \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix}.$$

This is called **scalar multiplication**. You can also write $\lambda \boldsymbol{u} = \boldsymbol{u} \lambda$, although writing the scalar first is usually preferred.

Careful! *This does not mean you can multiply a vector with another vector. This definition only allows you to multiply a vector by a scalar.*

Try multiplication by scalars out for yourself with this interactive graph.

Scalar multiplication works in such a way that a vector which is multiplied by a scalar keeps the same *direction*, but changes *magnitude*. This brings us to the following definition.

Definition 2.4. Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be **collinear** if there exists a scalar λ such that $\lambda \boldsymbol{u} = \boldsymbol{v}$. Collinear vectors either point in the same direction (if λ is positive) or in opposite directions (if λ is negative).

By convention, the zero vector $\boldsymbol{0}$ is collinear to every vector.

You can experiment with the idea of collinearity with this interactive graph.

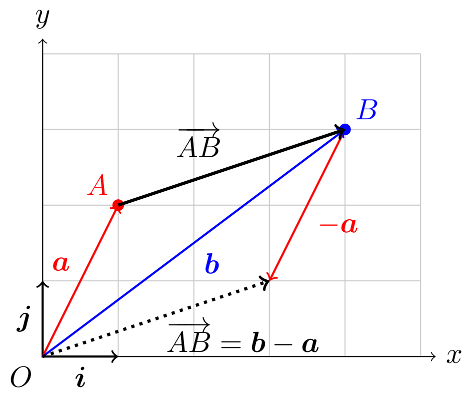
One last thing we can do now that we know how to add and multiply vectors is to see how to find the coordinates of a vector joining two points. Suppose you have a point $A(a_1, a_2, a_3)$ and a point $B(b_1, b_2, b_3)$ and you want to find the coordinates of the vector going from A to B, sometimes written $\overrightarrow{AB}$. This can be done using position vectors, which we saw in Subsection 1.3.

Let $\boldsymbol{a}$ be the position vector of A and $\boldsymbol{b}$ be the position vector of B. Then we want a vector which will lead us from the point A to the point B. We can do this by saying that we want to go from A to B *via* the origin O so that $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$. The vector $\overrightarrow{OB}$ is $\boldsymbol{b}$, and the vector $\overrightarrow{AO}$ is the opposite of $\overrightarrow{OA}$, the position vector of $\boldsymbol{a}$. So $\overrightarrow{AO} = -\boldsymbol{a}$. Therefore:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \boldsymbol{b} - \boldsymbol{a}. \tag{3}$$

Example 2.5. *In the xy-plane, let A have coordinates $(1, 2)$ and let B have coordinates $(4, 3)$. Represent the vector $\overrightarrow{AB}$ and find its coordinates.*

Answer. Let $\boldsymbol{a} = (1, 2)$ and $\boldsymbol{b} = (4, 3)$ denote the position vectors of A and B, respectively. In the plane, this is:



Using Equation 3, we get that:

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = (4,3) - (1,2) = (4-1, 3-2) = (3,1).$$

2.2 Finding the Norm of a Vector

We have mentioned several times that vectors have direction and magnitude, but without really explaining how to find the magnitude of a vector. This magnitude is usually called the **norm** of a vector. There are other ways of defining a norm, but the following definition is the only one that will be used in MT2501 and MT2503:

Definition 2.6. The (Euclidean) **norm** of a vector $\boldsymbol{u} = (u_1, u_2, u_3)$ is written $\|\boldsymbol{u}\|$ and is defined to be:

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

This measures the distance from the start of the vector to its end.

Careful! *The norm of a vector is a scalar quantity.*

Note that in higher and lower dimensions, this definition still holds - you need to change what's inside the square root so that every component of the vector is included. Therefore, the general formula for a vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ becomes:

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \tag{4}$$

See how the norm of a vector changes with this interactive graph.

Example 2.7. Find the norms of the following vectors:

$$\boldsymbol{a} = (3, -1, 2)\ ; \qquad \boldsymbol{b} = (4,3)\ ; \qquad \boldsymbol{0}\ ; \qquad \boldsymbol{i}.$$

Answer.

- $\|\boldsymbol{a}\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}.$

- $\|\boldsymbol{b}\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$.
- $\|\boldsymbol{0}\| = \sqrt{0^2 + 0^2 + 0^2} = 0$. The zero vector is in fact the only vector which has zero norm.
- $\|\boldsymbol{i}\| = \|(1, 0, 0)\| = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1$. This is consistent with what we have seen about unit vectors, as $\boldsymbol{i}$ is a unit vector which we said should have length 1.

Conveniently, this definition of a norm means that scaling a vector by λ also scales its norm by $|\lambda|$:

Theorem 2.8. *Let $\boldsymbol{u}$ be a vector and λ be a scalar. Then the following holds:*

$$\|\lambda \boldsymbol{u}\| = |\lambda| \, \|\boldsymbol{u}\| \, ,$$

where $|\lambda|$ denotes the absolute value of λ.

Before moving on to the next topic, let's look at something we can do with the norm of a vector: we can generate unit vectors pointing in any direction.

In a Cartesian frame of reference, we have three unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ which all point in different directions and have length 1. However, we may wish to create a unit vector of length 1 pointing in the direction of any combination of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. Consider some non-zero vector $\boldsymbol{u}$. If we want a unit vector pointing in the direction of $\boldsymbol{u}$, we can *normalise* it - that is, re-scale it by multiplying by some number - so that our new vector has length 1. We usually call this normalised unit vector $\hat{\boldsymbol{u}}$, and obtain it as follows:

$$\hat{\boldsymbol{u}} = \frac{1}{\|\boldsymbol{u}\|} \boldsymbol{u} \quad \text{since this implies} \quad \|\hat{\boldsymbol{u}}\| = \left| \frac{1}{\|\boldsymbol{u}\|} \right| \cdot \|\boldsymbol{u}\| = 1. \tag{5}$$

See this idea in action with this interactive graph.

2.3 The Dot Product

This next operation, which you may have seen before, appears frequently in various areas of mathematics, since it is a very good way of measuring the similarities between two vectors.

Definition 2.9. The **dot product** of two vectors $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, also called the **scalar product**, is written $\boldsymbol{u} \cdot \boldsymbol{v}$ and is defined as:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Careful! *The dot product combines two vectors together to form a scalar, not a vector.*

Just like the idea of a norm (Definition 2.6) can be extended to lower or higher dimensions, the dot product can be generalised as follows:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n. \tag{6}$$

The dot product is quite a "nice" operator as it behaves quite a lot like simple multiplication does:

Theorem 2.10. *Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be vectors. Then the following is always true:*

- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$;
- $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{v}) + (\boldsymbol{u} \cdot \boldsymbol{w})$;
- $\boldsymbol{u} \cdot \boldsymbol{u} = \|\boldsymbol{u}\|^2$.

The above results hold for vectors in any number of dimensions.

The dot product also links back to the geometric notion of vectors that has been discussed in Section 1. Suppose you have two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, at an angle of θ to each other. Then it can be shown that the scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta. \tag{7}$$

This equality holds regardless of the dimension you're working in - the angle between two vectors can always be found using this expression. This brings us to the following property:

Theorem 2.11. *Two vectors are orthogonal (at right angles with each other) if and only if their scalar product is zero.*

Applying the dot product to the Cartesian unit vectors therefore gives nice results:

$$\begin{aligned} \boldsymbol{i} \cdot \boldsymbol{i} = 1\,; \quad \boldsymbol{i} \cdot \boldsymbol{j} = 0\,; \quad \boldsymbol{i} \cdot \boldsymbol{k} = 0 \\ \boldsymbol{j} \cdot \boldsymbol{i} = 0\,; \quad \boldsymbol{j} \cdot \boldsymbol{j} = 1\,; \quad \boldsymbol{j} \cdot \boldsymbol{k} = 0 \\ \boldsymbol{k} \cdot \boldsymbol{i} = 0\,; \quad \boldsymbol{k} \cdot \boldsymbol{j} = 0\,; \quad \boldsymbol{k} \cdot \boldsymbol{k} = 1. \end{aligned} \tag{8}$$

Experiment with scalar products with this interactive graph.

2.4 The Cross Product

Unlike the other vector operations we have seen so far, the vector cross product *only* works in 3D - it's not defined if the vectors you're using have more, or fewer, components.

Definition 2.12. The **cross product** of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is defined as:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

Careful! *The cross product combines two vectors together to form another vector.*

For the purposes of memorisation, here's a way to think of the cross product which will save you from learning it all by heart. To find the x component of $\boldsymbol{u} \times \boldsymbol{v}$, all you need to do is multiply the two rows beneath the first one together diagonally, and subtract the red product from the blue one, like so:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} {\color{blue} u_2 v_3} - {\color{red} u_3 v_2} \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

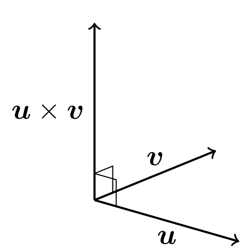
To get the second row of $\boldsymbol{u} \times \boldsymbol{v}$ from the first, change all the 1s to 2s, the 2s to 3s, and the 3s back to 1s. The same goes for getting the third row from the second.

If you already know how to find determinants of 3×3 matrices (see Chapter 10), you can also remember the cross product as the following determinant:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\boldsymbol{i} - (u_1 v_3 - u_3 v_1)\boldsymbol{j} + (u_1 v_2 - u_2 v_1)\boldsymbol{k}, \tag{9}$$

Where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the Cartesian unit vectors from Equation 2.

Now that we've seen how to remember this product, it's useful to get some geometric intuition for what it does! The vector cross product takes two three-dimensional vectors and produces a third vector which is always orthogonal to the other two; additionally, this third vector always points in a direction that follows the right-hand rule. Suppose $\boldsymbol{u}$ points in the x direction, and $\boldsymbol{v}$ points in the y direction in the right-hand rule; then $\boldsymbol{u} \times \boldsymbol{v}$ points in the direction of z.



Additionally, the length of the resulting vector $\boldsymbol{u} \times \boldsymbol{v}$ depends on the lengths of $\boldsymbol{u}$ and $\boldsymbol{v}$, in such a way that we obtain an equality similar to Equation 7:

$$\boldsymbol{u} \times \boldsymbol{v} = \underbrace{\underbrace{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \sin\theta}_{\text{scalar}} \; \underbrace{\hat{\boldsymbol{n}}}_{\text{vector}}}_{\text{vector}}, \tag{10}$$

where θ is the smallest angle between the two vectors (measured from $\boldsymbol{u}$ to $\boldsymbol{v}$), and $\hat{\boldsymbol{n}}$ is a unit vector pointing in a direction perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$, following the right-hand rule.

Now we can look at some properties of the cross product:

Theorem 2.13. *Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be vectors, and let $\mathbf{0}$ be the zero vector. Then the following properties are always true:*

- $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$;
- $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{w})$;
- $\boldsymbol{u} \times \boldsymbol{u} = \mathbf{0}$.

Careful! *The cross product is not associative, so $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$ is not always true.*

Because the cross product is linked with the right-hand rule, it is not surprising that it gives nice results when applied to the Cartesian unit vectors (you can calculate these yourself using the coordinates of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ from Equation 2):

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}; \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}; \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}. \tag{11}$$

One other thing the cross product is useful for is finding the area of a parallelogram. If you have a parallelogram with sides corresponding to vectors $\boldsymbol{u}$ and $\boldsymbol{u}$, then the area of this shape is the magnitude of the cross product of the two vectors:

$$A_{\text{parallelogram}} = \|\boldsymbol{u} \times \boldsymbol{v}\|. \tag{12}$$

3 Cartesian Geometry

3.1 Definition

In this final section, we will look at ways in which we can describe shapes in 2D and 3D space using Cartesian equations.

To do this, we must first settle on what exactly we mean by a "shape". In this guide, we will be looking at "shapes" such as planes, circles and spheres. Let's suppose we're looking at a plane, called S, in 3D space. If you've seen set theory before (see Chapter 1), you could say that if $\mathbb{R}^3$ is the set of all points in the 3D space, and if S is the set of all points on the plane, then $S \subset \mathbb{R}^3$: S is a subset of $\mathbb{R}^3$.

Usually, a shape like S is referred to not as a *shape*, but as a **locus**. Our goal in this section is, to find a way of describing a given locus: a way of telling which points are part of the locus, and which points are not. We can use Cartesian equations for this.

Definition 3.1. Let S be any locus in 3-dimensional space with coordinates x, y, z.

A **Cartesian equation** of S is an equation of the form

$$f(x, y, z) = 0,$$

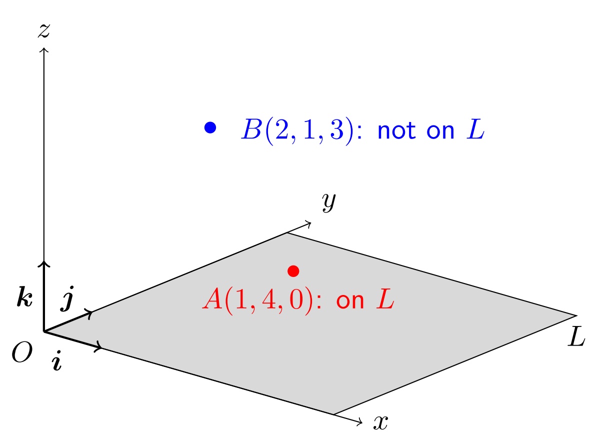
where f is some function of x, y and z, chosen so that the set of solutions (x, y, z) to this equation corresponds to the set of points on S.

All this means is that we can describe geometric shapes, or *loci*, by writing out an equation which is only satisfied by the points of the shape.

Note that this definition is for three dimensions, but can be changed to work in lower or higher dimensions - for example, the Cartesian equation of a locus S in two dimensions is $f(x, y) = 0$.

Example 3.2. *Find a Cartesian equation describing the locus S corresponding to the xy-plane in 3D space.*

Answer. We want to find an equation $f(x, y, z) = 0$ which is satisfied by all the points of the xy-plane, and only by these points. One characteristic of the xy-plane is that it contains all the points that have a zero z-coordinate:



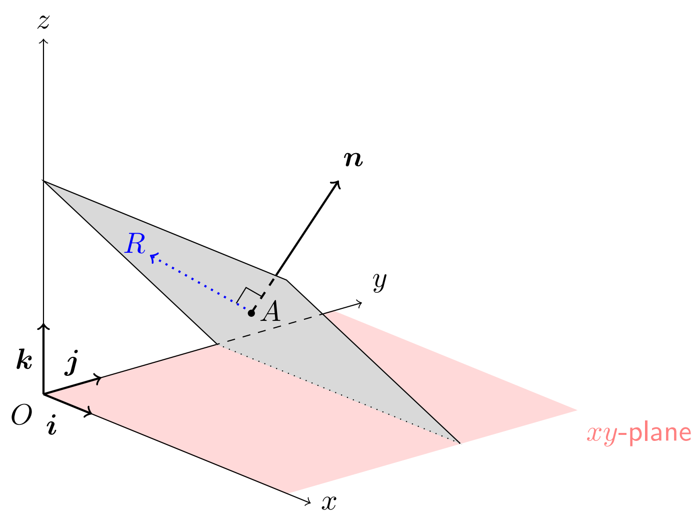
This is a Cartesian equation for the xy-plane: $f(x, y, z) = z = 0$.

Now we can look at a more general way of finding the equation of any plane.

3.2 Planes

In 3D space, one way a plane can be constructed is with a point and a normal vector to the plane:

Definition 3.3. Let A be a point and $\boldsymbol{n}$ be a vector. The set of all points R such that $\overrightarrow{AR} = \boldsymbol{r} - \boldsymbol{a}$ and $\boldsymbol{n}$ are *orthogonal* (at right angles) is a plane containing A and normal to (perpendicular to) $\boldsymbol{n}$.



We're going to use this definition to find the Cartesian equation of a plane.

Let A be a point with coordinates (a_1, a_2, a_3), and let $\boldsymbol{n} = (n_1, n_2, n_3)$. Call $\mathscr{P}$ the plane containing A and normal to $\boldsymbol{n}$. Now suppose $R(x, y, z)$ is on $\mathscr{P}$. From Definition 3.3, this implies that $\overrightarrow{AR}$ and $\boldsymbol{n}$ are orthogonal. So by Theorem 2.11, $\overrightarrow{AR} \cdot \boldsymbol{n} = 0$. This is equivalent to:

$$\begin{aligned} & (\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0 \\ \iff & \boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} && \text{(by Theorem 2.10)} \\ \iff & n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3 \\ \iff & n_1 x + n_2 y + n_3 z + d = 0, \end{aligned}$$

where $d = -(a_1 n_1 + a_2 n_2 + a_3 n_3)$. This is in the form of a Cartesian equation:

Theorem 3.4. *A Cartesian equation for the plane normal to* $\boldsymbol{n} = (n_1, n_2, n_3)$ *and containing the point* $A(a_1, a_2, a_3)$ *is:*

$$n_1 x + n_2 y + n_3 z + d = 0,$$

where $d = -(a_1 n_1 + a_2 n_2 + a_3 n_3)$.

This also works the other way around: if a plane has Cartesian equation $Ax + By + Cz + D = 0$, then a normal vector to the plane is the vector (A, B, C).

With this theorem in mind, if you are asked to find the Cartesian equation of a plane containing a certain point and perpendicular to a certain vector, you can either plug the numbers into the formula above, or work out the equation from the beginning (starting with $(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$).

Note that you can also find the equation of a plane if instead of being given a point and a vector, you're given three distinct points P, M and N which lie in the plane. In this case, consider the vectors $\overrightarrow{PM}$ and $\overrightarrow{PN}$, which are aligned with the plane: their cross product is perpendicular to both vectors and is therefore perpendicular to the plane. So you can find the Cartesian equation of such a plane using the coordinates of the normal vector $\overrightarrow{PN} \times \overrightarrow{PM}$.

Describing planes with a normal vectors also allows us to know when two planes are parallel.

Theorem 3.5. *Two planes are parallel (and possibly equal) if and only if the vectors orthogonal to these planes are collinear.*

3.3 Circles

We can also use Cartesian equations to describe curved surfaces. One example you may have seen before is the circle.

Suppose we are trying to find a Cartesian equation in 2D for a circle $\mathscr{C}$, which has centre $A(a_1, a_2)$ and radius $r > 0$. $\mathscr{C}$ is the set of all points at a distance of r from A: using the formula for Euclidean distance, we get that the distance d of some point $R(x, y)$ from the point A is:

$$d = \sqrt{(x - a_1)^2 + (y - a_2)^2}. \tag{13}$$

In this case, we want the distance to equal the radius r, so our equation becomes

$$\begin{aligned} &\sqrt{(x - a_1)^2 + (y - a_2)^2} = r \\ \iff &(x - a_1)^2 + (y - a_2)^2 = r^2, \end{aligned}$$

since both sides are non-negative. This brings us to our next theorem:

Theorem 3.6. *A Cartesian equation for the circle with centre $A(a_1, a_2)$ and with radius $r \geq 0$ is:*

$$(x - a_1)^2 + (y - a_2)^2 = r^2.$$

Note that often, when studying a circle, we consider its centre to be at the origin so that the equation simplifies to: $x^2 + y^2 = r^2$.

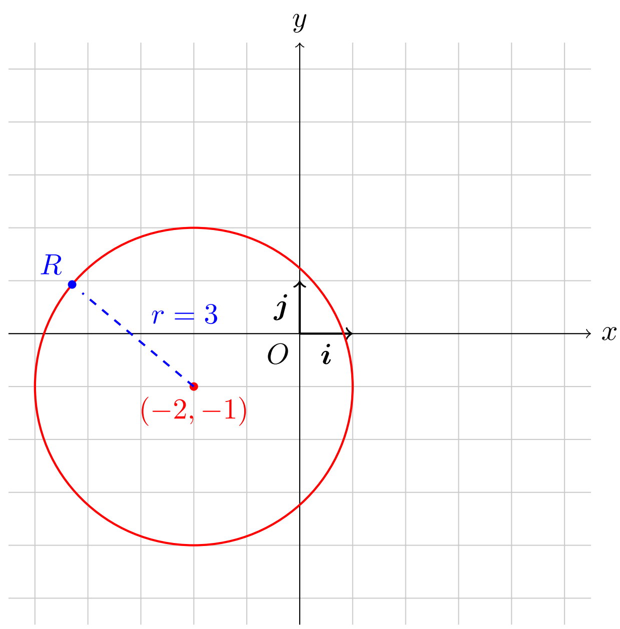
Example 3.7. *In the xy-plane, draw the locus defined by the following Cartesian equation:*

$$x^2 + 4x + y^2 + 2y = 4.$$

Answer. It may not be obvious at first that this equation defines a circle, but the fact that x and y are squared in the question might make us wonder if this is not the case. If we want to show that this equation corresponds to a circle, we need to manipulate it back into a form which looks like the one in Theorem 3.6. To do this, we can use a common technique called **completing the square**: We force the variables into factorised form by adding and subtracting the same number, like so: $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1$. In this case, we get:

$$\begin{aligned} x^2 + 4x + y^2 + 2y &= 4 \\ (x^2 + 4x + 4) + y^2 + 2y &= 8 \\ (x^2 + 4x + 4) + (y^2 + 2y + 1) &= 9 \\ (x - (-2))^2 + (y - (-1))^2 &= 3^2. \end{aligned}$$

This is in the correct form, and we deduce from Theorem 3.6 that the locus is a circle with centre $(-2, -1)$ and with radius 3:



Have a look at Cartesian equations of circles with this interactive graph.

3.4 Ellipses

From the Cartesian equation of a circle comes its natural extension: the ellipse. An ellipse is a circle but stretched horizontally and vertically according to some factors a and b, which must therefore also be part of its Cartesian equation:

Theorem 3.8. *The Cartesian equation of an ellipse with centre $A(a_1, a_2)$, horizontal radius a and vertical radius b ($a, b > 0$) is:*

$$\frac{(x-a_1)^2}{a^2} + \frac{(y-a_2)^2}{b^2} = 1.$$

The smaller of a and b is commonly called the **semi-minor axis** *of the ellipse; the greater is called the* **semi-major axis**.

Once again, if given an expression with x^2 and y^2 terms, you may find when factoring it that it is the equation of an ellipse. Usually, if the coefficients of x^2 and y^2 are the same, the locus described is a circle; if the coefficients are different, the locus is an ellipse.

This interactive graph lets you see how an ellipse changes as a_1, a_2, a and b change.

3.5 Hyperbolae

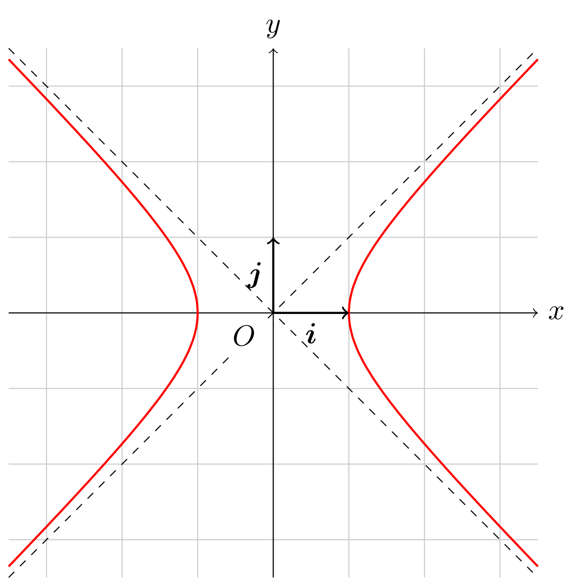
The next shape we will look at is the hyperbola, which has a very similar equation to those of the circle and ellipse. This shape appears a lot in Cartesian geometry but also sometimes in calculus, as it is linked to the $\cosh$ and $\sinh$ functions as well as to the graph of $f : x \mapsto \frac{1}{x}$.

Definition 3.9. A **hyperbola** is a shape defined by the Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a, b > 0$.

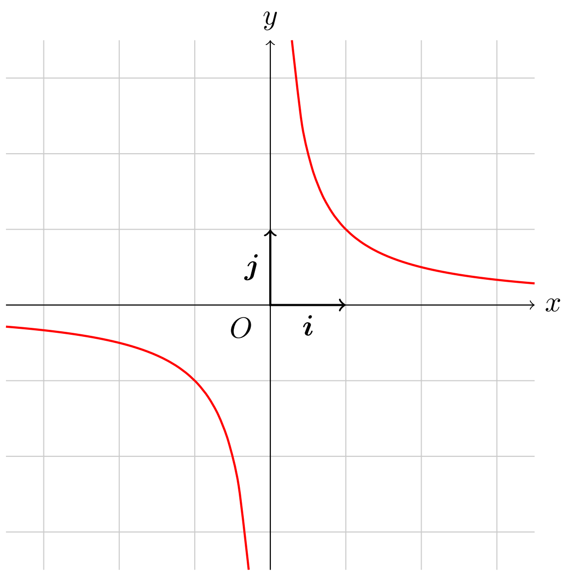
This shape is symmetric about the origin O, and as you get further from the origin the lines flatten out and have asymptotes $y = \pm\frac{b}{a}x$ (see **Chapter 6** on function sketching). The following figure represents the *unit hyperbola* - when $a = b = 1$.



See this interactive graph to try out different hyperbola equations.

Note: Not all hyperbolae can be described using the equation in Definition 3.9. The equation in this definition will only produce hyperbolae that are symmetric about the x and y axes, but any 2D locus having the same shape, even if it is rotated or moved so that the centre is no longer the origin O, is still considered to be a hyperbola - it just can't be described using the equation above.

One example that you might see is the graph of the function $f : x \mapsto \frac{1}{x}$ (below, left). This is also a hyperbola, but rotated 45 degrees clockwise relative to the one in the image above. Another example is the hyperbola with equation $y^2 - x^2 = 1$ (below, right). Here the formula is similar but the x and y have been reversed, resulting in a reflection of the shape through the line $y = x$.



Shifting the hyperbola so that its centre is no longer the origin O requires changing the equation slightly, so that it becomes:

$$\frac{(x - a_1)^2}{a^2} - \frac{(y - a_2)^2}{b^2} = 1. \tag{14}$$

The hyperbola described by this equation has centre $A(a_1, a_2)$.

3.6 Parabolae

A somewhat similar locus to the hyperbola is the parabola. This may be more familiar, as the parabola arises in calculus in the form of the graph of any quadratic function. Unlike the hyperbola, it does not have any asymptotes - nor does it have two disjoint symmetric parts.

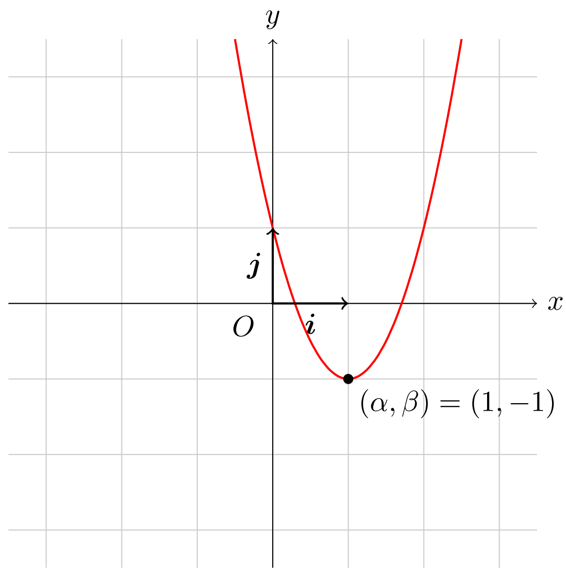
Definition 3.10. A **parabola** is the shape defined by the Cartesian equation

$$y = a(x - \alpha)^2 + \beta,$$

Where α, β and a are fixed real numbers. The parabola has **vertex** (α, β) and is symmetric about the line $x = \alpha$. The factor a controls the horizontal scaling of the shape.

Expanding the $(x-\alpha)^2$ term changes the equation into the well-known quadratic form $y = ax^2+bx+c$, where $\alpha = -\frac{b}{2a}$ and $\beta = a\alpha^2 + b\alpha + c$. This is indeed the graph of a quadratic function.

Below is a representation of the parabola $y = 2x^2 - 4x + 1 = 2(x-1)^2 - 1$.



Once again, there exist parabolae which cannot be described using the Cartesian equation from Definition 3.10: for example, rotated figures such as $x = y^2$ are still considered parabolae.

3.7 Spheres

The final locus that we will look at in this section is the sphere - unlike for the circle, ellipse and hyperbola, we are now in three-dimensional space. The sphere is the generalisation of a circle to 3D space, and therefore corresponds to the set of points which are at a distance $r > 0$ from a point $A(a_1, a_2, a_3)$ which is the centre of the sphere. Just like with our construction of a circle, we can express the distance between A and a point $R(x, y, z)$ as $\sqrt{(x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2}$. Therefore, we get the following equation:

Theorem 3.11. *A Cartesian equation of a sphere centred on a point $A(a_1, a_2, a_3)$ and with a radius $r > 0$ is:*

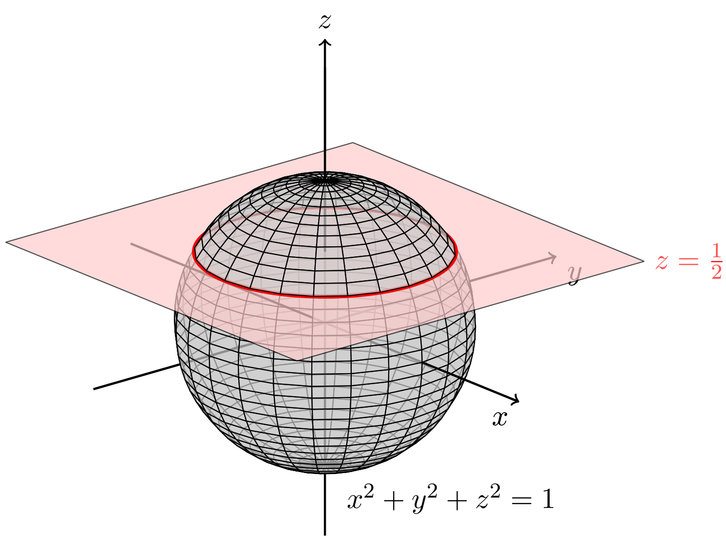
$$(x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2 = r^2.$$

Example 3.12. *Find the Cartesian equation of the locus of intersection of the unit sphere and the plane* $z = \frac{1}{2}$.

Answer. Here we are given two surfaces and asked to describe how they intersect. The *unit sphere* is the sphere centred at the origin with radius 1: so it has Cartesian equation $x^2 + y^2 + z^2 = 1$. We are looking for points in 3D space which satisfy both Cartesian equations:

$$x^2 + y^2 + z^2 = 1 \qquad (*)$$
$$z = \frac{1}{2} \qquad (**)$$

Substituting $(**)$ into $(*)$ gives $x^2 + y^2 + \left(\frac{1}{2}\right)^2 = 1$, which is equivalent to $x^2 + y^2 = \frac{3}{4}$. This is the equation of a circle centred at the origin in two-dimensional space (see Theorem 3.6), with radius $\sqrt{\frac{3}{4}}$. Additionally, this circle is at height $z = \frac{1}{2}$, according to the second equation.



So, we have a system of Cartesian equations:

$$\begin{cases} x^2 + y^2 = \left(\sqrt{\frac{3}{4}}\right)^2 \\ z = \frac{1}{2}. \end{cases}$$

Although it does not quite match the form of the Cartesian equation outlined in Definition 3.1, this equation is still considered a Cartesian equation as it imposes a restriction on the three variables x, y and z in such a way that a locus is determined.